PHYSICAL REVIEW E VOLUME 47, NUMBER 1 JANUARY 1993

# Nonstable solitons and sharp criteria for wave collapse

S. K. Turitsyn*
*Institut für Theoretische Physik I, Heinrich-Heine-Universität Düsseldorf, D-4000 Düsseldorf 1, Germany*

(Received 15 October 1992)

Sharp sufficient criteria for collapse are found for the nonlinear Schrödinger equation in the so-called supercritical case as well as for the Ginzburg-Landau equation in the case of the subcritical bifurcation. It is demonstrated that nonstable solitons in these models, under some additional assumptions, play the role of a "boundary" (saddle points) between collapsing and noncollapsing solutions.

PACS number(s): 05.45.+b, 52.35.Sb, 52.35.Mw, 03.40.Kf

## I. INTRODUCTION

As is well known, there exists a correlation between the stability of stationary solutions of a nonlinear equation and the possibility of wave collapse. In most cases, the instability of steady-state solutions of a nonlinear model is accompanied by the existence of blow-up for a certain class of initial data. Wave collapse is a classical example of a very nonlinear phenomenon that occurs in different physical contexts (see, e.g., for a review [1–4]). However, only for a few models, which are of interest for applications, can the existence of the blow-up be proved analytically. Our motivation in the present paper is to establish a link between the instability of soliton solutions in a nonlinear system and the sharp criteria of the blow-up on the example of two of these models.

Before we present our results, we briefly review the basis features of some known nonlinear systems for which sufficient criteria of the blow-up may be found analytically (see for the details [1–4] and references therein), in order to provide a basis for our subsequent discussion.

*(1) The nonlinear Schrödinger equation:*

$$i\Psi_t = -\Delta\Psi - (\sigma+1)|\Psi|^{2\sigma}\Psi = \frac{\delta H}{\delta \Psi^*}\ , \tag{1}$$

where the Hamiltonian $H$ is of the form $H=\int|\nabla\Psi|^2 d\mathbf{r} - \int|\Psi|^{2\sigma+2}d\mathbf{r}$. We consider in this paper Eq. (1) for the supercritical case, defined by the inequality $\sigma d>2$.

*(2) Nonlinear heating equation:*

$$U_t = \Delta U + U^{1+\alpha}\ , \tag{2}$$

where the coefficient $\alpha$ is supposed to be positive. There exist many generalizations of this equation. For simplicity, we mention here only the basic model.

*(3) Modified Kadomtsev-Petviashvili equation,* describing the propagation of the narrow beams of the nonlinear waves in weakly dispersive media. In particular, the evolution of the magnetoelastic waves in antiferromagnets [5] is governed by Eq. (3):

$$\frac{\partial U}{\partial t} = -(n+2)(n+1)U^n U_x - U_{xxx} + \Delta_\perp W = \frac{\delta H}{\delta U},\quad W_x = U\ . \tag{3}$$

For this equation the Hamiltonian $H$ is defined as

$$H=\int[\tfrac{1}{2}U_x^2+\tfrac{1}{2}(\nabla_\perp W)^2 - U^{n+2}]d\mathbf{r}\ . \tag{4}$$

Here $n$ is supposed to satisfy the inequality $n\geq(d+1)/(d-1)$, $d$ is a space dimension.

*(4) The well-posed Boussinesq equation:*

$$U_{tt} - U_{xx} + U_{xxxx} + (U^2)_{xx} = 0\ . \tag{5}$$

Equation (5) may be written in the Hamiltonian form

$$U_t = -\frac{\delta H}{\delta \Phi},\quad \Phi_t = \frac{\delta H}{\delta U}\ . \tag{6}$$

For the system (6) the Hamiltonian $H$ is of the form

$$H=\int(\tfrac{1}{2}\Phi_x^2+\tfrac{1}{2}U^2+\tfrac{1}{2}U_x^2-\tfrac{1}{3}U^3)dx\ ,$$

Although, the models mentioned above describe rather different physical problems, they have much in common. In particular, in each of them negativeness of some functional $H$ is a sufficient condition (in the general case, under some additional assumptions) of blow-up. Equations (1), (3), and (4) are examples of the Hamiltonian systems. The sufficient criterion of the collapse in Eqs. (1) and (3) has been found to be $H\leq 0$ [see Refs. [6,7] for Eq. (1) and Ref. [5] for Eq. (2)], where by $H$ one means the corresponding Hamiltonian.

For the Boussinesq equation the sufficient condition of blow-up is $H\leq 0$ under additional assumptions [8]: (a) $\int f\Phi_x dx|_{t=0}>0$, $f_x=U$ and (b) $\int U\,dx=0$.

The nonlinear heating equation (or nonlinear diffusion equation) is an example of a dissipative system. For Eq. (2) there exists a kind of Lyapunov functional

$$H=\int\left[\tfrac{1}{2}(\nabla U)^2 - \frac{1}{2+\alpha}U^{2+\alpha}\right]d\mathbf{r}\ ,$$

satisfying the following evolution equation:

$$\frac{dH}{dt} = -\int U_t^2 d\mathbf{x}\ . \tag{7}$$

If initial data are chosen in such a way that the inequality $H|_{t=0}=<0$ holds, then solutions of Eq. (2) become singular in a finite time [9].

The main intent of the present work is to demonstrate that such criteria may be sharpened. Sharp sufficient conditions for the blow-up will be obtained for two known nonlinear models. A general statement, concerning the link between unstable solitons and the criterion for the blow-up in the arbitrary nonlinear system, will be formulated as a hypothesis.

It is well known that for many nonlinear models, interplay between nonlinearity and dispersion results in a special class of stationary solutions, namely *solitons*. A soliton corresponds to an exact balance between nonlinear and dispersion effects. In the case of stable equilibrium, the corresponding solitons are stable against small perturbations. The stable solitons play a fundamental role in the dynamics of nonlinear systems. Often, however, the balance between nonlinearity and dispersion corresponds to unstable stationary solutions. It means that, although for the stationary problem the influence of nonlinear effects may be compensated by dispersion broadening of the wave packet, small perturbations around the soliton will break such an equilibrium. That leads to the instability of the corresponding solitons. As was mentioned above, there exists the link between the instability of the solitons and wave collapse. In most cases the instability of the solitons in nonlinear systems is associated with the existence of the blow-up in such models. The term "blow-up" is generally used to refer to the nonexistence of the initial-value problem for all time. As will be shown in the present paper, nonstable solitons play an important role in the determination of the sharp criteria for the collapse.

The basic idea is to use the fact that solitons appear as a result of the balance between forces that control the dynamical evolution of the nonlinear system. In other words, nonstable solitons may be viewed as solutions lying on the boundary between solutions whose evolution is determined mainly by dispersion effects and those for which nonlinear effects are predominant. It is natural to presuppose that the evolution of the solutions, for which nonlinear effects dominate, leads to the blow-up, if the balance between nonlinearity and disperison is unstable.

Using two well-known nonlinear equations, the nonlinear Schrödinger equation and the Ginzburg-Landau equation, as examples, we will demonstrate that nonstable soliton solutions, under some additional assumptions, play the role of a "boundary" between collapsing and noncollapsing regimes. This statement may be suggested as a hypothesis for any nonlinear system that demonstrates the blow-up-type dynamics and possesses unstable soliton solutions.

## II. THE NONLINEAR SCHRÖDINGER EQUATION IN THE SUPERCRITICAL CASE

In this section we shall illustrate the main idea on the example of the nonlinear Schrödinger (NLS) equation, which has become one of the basic models in nonlinear science. A review of many aspects of this model has been given, e.g., in [1–4]. The NLS equation with the power nonlinearity has nonstable soliton solutions in the so-called supercritical case: $2/D<\sigma<2/(D-2)$.

For the critical case $\sigma=2/D$ the sharp sufficient condition for the blow-up was obtained by Weinstein [10]. For simplicity (but without loss of generality) we consider the case $\sigma=1$ and $D=3$. In this case, for example the NLS equation occurs as the subsonic limit of the Zakharov equations, describing Langmuir waves in plasma. The analysis of the general case will be presented elsewhere [11].

From the well-known relation

$$\frac{\partial^2}{\partial t^2}\int r^2|\psi|^2 dr=8H-4I_2 \tag{8}$$

(where the Hamiltonian $H=\int|\nabla\Psi|^2 d\mathbf{r}-\int|\Psi|^{2\sigma+2}d\mathbf{r}=I_1-I_2$) it is fairly obvious that the negativity of the Hamiltonian gives a sufficient condition for collapse, but, at the same time, this condition is not sharp because in the right-hand side of this relation there is the additional *negative* term $-4I_2$. It is evident that if we find a way to estimate this term from below by some constant (depending on integrals of motion) then we will get a more sharp sufficient criterion for collapse. In order to get such an inequality, we show first that the Hamiltonian can be bounded from below by some function of $I_2$. To do this, one may use the following inequality:

$$I_1\geq C_0N^{-3}I_2^{2/3}\ ; \tag{9}$$

here $N=\int|\psi|^2 dr$ and the best interpolation constant $C_0=\frac{3}{4}N_{\rm cr}^2$ was calculated by Weinstein [10]. The value $N_{\rm cr}$ is the value of the integral of motion $N$ calculated for the ground-state solution of the equation: $\Delta f-f+f^3=0$.

Substituting this estimation into the Hamiltonian we get the following inequality:

$$H\geq F(I_2)=C_0N^{-3}I_2^{2/3}-I_2\ . \tag{10}$$

The remarkable fact is that the maximum of the function $F(I_2)$ is equal to $\max\{F(I_2)\}=H_s$ *exactly*, where $H_s$ is the value of the Hamiltonian for the soliton solution, and this maximum is attained at the point $I_2=I_{2s}=2H_s$, $I_{2s}$ being the value of the integral $I_2$ calculated for the soliton.

It follows that for $0<H<H_s$ the equation $H=F(I_2)$ has two solutions, $I_2^{(-)}$ and $I_2^{(+)}$. Inequality (16) is satisfied for two intervals $0\leq I_2\leq I_2^{(-)}$ and $I_2\geq I_2^{(+)}$. Our basic equation implies that any initially smooth solution evolves continuously in the time interval of the existence, hence the integral $I_2(t)$ is continuous for any $t$ before the collapse point. Since the intervals $(0,I_2^{(-)})$ and $(I_2^{(+)},\infty)$ are disjoint, $I_2(t)$ can never cross from one interval into another. Now suppose that at the initial moment $I_2|_{t=0}>I_{2s}$, then $I_2(t)>I_{2s}$ for all $t$. Substituting the estimate for $I_2(t)$ into Eq. (8) we obtain the main formula for this section:

$$\frac{\partial^2}{\partial t^2}\int r^2|\psi|^2 dr\leq 8(H-H_s)\ . \tag{11}$$

Thus, the sharp sufficient criterion for the blow-up dynamics is the condition $H<H_s$. It should be noted that in the present paper, in order to avoid unessential complexity, we limit our consideration to the case in which there is no special focusing on the initial moment

($\partial/\partial t \int r^2|\psi|^2 dr = 0$). The more general case will be considered elsewhere.

We may conclude that the solutions, for which the Hamiltonian is less than the value of $H_s$ calculated on the soliton solution *(nonstable)*, become singular in a finite time, so that the nonstable soliton solutions in some sense play the role of a "boundary" between collapsing and noncollapsing regimes (for details see [11]).

## III. THE GINZBURG-LANDAU EQUATION

In this section a similar theorem will be proved for the Ginzburg-Landau equation in case of subcritical bifurcation.

We write the Ginzburg-Landau (GL) equation in the following form:

$$\frac{\partial S}{\partial t} = -aS + b\frac{\partial^2 S}{\partial x^2} + 2c|S|^2 S , \tag{12}$$

with the boundary condition $S \to 0$ as $|x| \to \infty$. Equation (12) results from an expansion in some physical parameter near the critical point. In the general case, coefficients $a$, $b$, and $c$ are complex. Because of the generality, the GL equation may be applied to a variety of physical problems, describing the slow evolution of a mode that bifurcates via an oscillatory instability from a homogeneous basic state (see, e.g., Refs. [12–15]). For the case of subcritical bifurcation [$\mathrm{Re}(c) > 0$], the GL equation may be used to describe plane shear-flow instabilities, binary-fluid convection in $a$ the subcritical range, etc. (see, e.g., references in [15]).

Equation (12) with the real coefficients describes, in particular, a distributed system, modeling dynamics of a pulse envelope propagating through a coupled-cavity mode-locked laser, for a phase-mismatch angle of $\pi/2$ between the two cavities [16].

Throughout this paper we assume coefficients $a$, $b$, $c$ to be real and positive. We begin by summarizing the main properties of Eq. (12).

A trivial homogeneous solution of the Eq. (12), $S = 0$, is stable against small perturbations, provided the coefficient $a$ is positive. Thus, under sufficiently weak initial disturbances the bound state ($S = 0$) remains stable and the disturbances die out. However, for the initial data for which nonlinearity is sufficient enough the situation changes. A perturbation will experience explosive growth when it becomes sufficiently large for the nonlinear term to come into play. Then the cubic term in Eq. (12) dominates, causing an amplification of the rate of increase of the maximum value of $|S|$, a focusing phenomenon occurs, and $S$ becomes singular at some finite point $t_1$.

The fundamental issue is that the dynamical evolution of the localized wave packet in Eq. (12) is determined by the interplay between nonlinear and diffusion terms in the right-hand side of Eq. (12).

As a result of the balance between nonlinearity and diffusion there exists an exact soliton solution of Eq. (12) in the following form:

$$S(t,x) = S_0 \mathrm{sech}(x/L) , \tag{13}$$

where the pulse width is given by $L = \sqrt{b/a}$ and the soliton amplitude is $S_0 = \sqrt{a/c}$. However, this balance is unsteady and the soliton pulse is unstable (see, e.g., [17]), i.e., it means that, although the interplay between dispersion and nonlinearity allows one to construct a steady-state solution, there exist perturbations of the soliton pulse for which nonlinear effects dominate and a collapse may appear for some class of the initial distributions.

A sufficient criterion for the wave collapse (a class of initial distributions leading to the collapse dynamics) was found in [18] by using the method of a majoring equation [2,19]. In the present paper we find a sharp criterion for the blow-up in the GL equation (12).

Consider two functionals $Q$ and $P$ defined by

$$Q = \int (b|S_x|^2 + a|S|^2 - c|S|^4)dx = Y + aP - I \tag{14}$$

and

$$P = \int |S|^2 dx . \tag{15}$$

The first functional may be viewed as the Lyapunov functional for the infinite-dimensional system (12), because $Q$ obeys the following equation:

$$\frac{dQ}{dt} = -2\int |S_t|^2 dx . \tag{16}$$

It means that one possible limiting asymptotic for Eq. (12) is $S \to 0$ as $t \to \infty$. If $S$ is sufficiently small at the point $t = 0$, this limit will be achieved.

Differentiating $P$ with respect to $t$, we get the equation

$$\frac{d}{dt}P = -2b\int |S_x|^2 dx + 2a\int |S|^2 dx + 4c\int |S|^4 dx$$
$$= -2Q + 2c\int |S|^4 dx . \tag{17}$$

It follows from Eq. (17) that $(d/dt)P > 0$ for the $Q < 0$. Because $Q$ is the nonincreasing function of $t$, if the latter requirement is satisfied at the initial moment of time, it will hold for all $t$. Evidently, if we prove that $\int |S|^4 dx$ may be bounded from below by some quantity, which does not depend on time, then we obtain a more sharp sufficient condition for increasing of the functional $P$.

We demonstrate first that the functional $Q$ may be bounded from below by some function of $I = c\int |S|^4 dx$.

Using the inequality (see, e.g., [10])

$$\int |S|^4 dx \le \frac{1}{\sqrt{3}} \left[ \int |S|^2 dx \right]^{3/2} \left[ \int |S_x|^2 dx \right]^{1/2} , \tag{18}$$

we have

$$Q = Y + aP - I \ge 3\frac{b}{c^2}\frac{I^2}{P^3} + aP - I \ge 4\left[\frac{a^3 b}{9c^2}\right]^{1/4} I^{1/2} - I$$
$$= 2Q_s^{1/2} I^{1/2} - I = f(I) . \tag{19}$$

Here $Q_s$ is a value of the integral $Q$ calculated on the soliton solution (13). Note that the function $f$ has a unique maximum which is *exactly equal* to $Q_s$. This maximum is attained at the point $I = I_s = Q_s$. So that, if at the moment $t = 0$ $Q < Q_s$ and additionally $I_0 > I_s$, then $I(t) > I_s = Q_s$ for any $t > 0$.

To find a sharp criterion for the blow-up in the case of Eq. (12), we need also the following inequality, which is valid for $I > Q_s$:

$$Y + aP - 2Q_s \geq 3\frac{b}{c^2}\frac{I^2}{P^3} + aP - 2Q_s$$
$$\geq 3\frac{b}{c^2}\frac{Q^2_s}{P^3} + aP - 2Q_s$$
$$\geq 4\left[\frac{a^3 b}{oc^2}\right]^{1/4} Q_s^{1/2} - 2Q_s = 0 \ . \quad (20)$$

Suppose now that the initial value $S$ is chosen so that the following conditions hold: (a) $Q_0 < Q_s$, and (b) $I_0 > I_s = Q_s$. Then solutions of Eq. (12) can exist only for a finite time. To prove this statement, we show that some integral characteristic of solutions that is a positive growing function of $t$ (the majoring function) tends to infinity as $t$ tends to some fixed $t_1$. Consider the time evolution of the following quantity:

$$R = \frac{Q_s - Q}{P} \ . \quad (21)$$

Differentiating $R$ with respect to $t$, we obtain

$$\frac{d}{dt}R = 2P^{-1}\int |S_t|^2 dx - \frac{(Q_s - Q)P_t}{P^2} \ . \quad (22)$$

The first term in Eq. (22) can be estimated by using the Hölder inequality

$$\frac{d}{dt}P = 2\int |S|\,|S|_t dx$$
$$\leq 2\left(\int |S|^2 dx\right)^{1/2}\left(\int |S_t|^2 dx\right)^{1/2} ,$$

in combination with the formula (17). Inserting the estimate for $\int |S_t|^2 dx$ into Eq. (22), we get the inequality

$$\frac{dR}{dt} \geq \frac{P_t}{P^2}\left[\frac{P_t}{2} + Q - Q_s\right] \geq 2\frac{R}{P}(Q_s - Q + Y + aP - 2Q_s)$$
$$\geq 2R^2 \ . \quad (23)$$

Integrating inequality (23) over time, we find

$$\int_{R_0}^{R} \frac{dR}{R^2} = \frac{1}{R_0} - \frac{1}{R} \geq 2t \ ,$$

i.e., rewriting this formula, we obtain, finally the estimate of the asymptotic form of the function $R$:

$$2R \geq \frac{1}{(t_1 - t)} \ .$$

It follows from the last expression that at the point $t_0 \leq t_1 = 1/2R_0$ the function $R(t)$ becomes infinite. We conclude from this that, if the initial condition is sufficiently nonlinear, so that nonlinear term in $Q$ dominates, the perturbations grow indefinitely [within the framework of Eq. (12)] in a finite time. The sharp sufficient condition for the blow-up in Eq. (12) is given by

$$Q|_{t=0} < Q_s = \tfrac{4}{3}a^{3/2}b^{1/2}c^{-1}$$

and

$$I_2 = c\int |S|^4 dx > Q_s \ .$$

A good feature of the method that we used is that it generalizes obviously to any space dimension.

In conclusion, we analyze in this paper the role played by nonstable solitons in the dynamics of nonlinear systems. Based on the well-known fact that the soliton occurs as a result of the balance between forces responsible for the dynamical evolution of the localized wave packet in the nonlinear system, we suggest that nonstable solitons lie on the "boundary," separating collapsing and noncollapsing solutions. In this way we found sharp criteria for the blow-up for the nonlinear Schrödinger equation in the supercritical case and for the Ginzberg-Landau equation in the case of subcritical bifurcation. The analytical method used here may be applied to other nonlinear systems.

---

*Permanent address: Institute of Automation and Electrometry, Russian Academy of Sciences, Novosibirsk 630090, Russia.

[1] V. E. Zakharov, in *Handbook of Plasma Physics,* edited by M. N. Rosenbluth and R. Z. Sagdeev (Elsevier, Amsterdam, 1984).
[2] H. A. Levine, SIAM Rev. **32**, 283 (1990).
[3] J. J. Rasmussen and K. Rypdal, Phys. Scr. **33**, 481 (1986).
[4] D. W. McLaughlin, G. Papanicolaou, C. Sulem, and P. L. Sulem, Phys. Rev. A **34**, 1200 (1986); N. E. Kosmatov, V. F. Shvets, and V. E. Zakharov, Physica D **52**, 16 (1991).
[5] S. K. Turitsyn and G. E. Fal'kovich, Zh. Eksp. Teor. Fiz. **89**, 258 (1985) [Sov. Phys. JETP **62**, 146 (1985)].
[6] V. N. Vlasov, I. A. Petrischev, and V. I. Talanov, Izv. Vys. Uchebn. Zaved. Radiofizika **14**, 1352 (1971).
[7] V. E. Zakharov, Zh. Eksp. Teor. Fiz. **62**, 1745 (1972) [Sov. Phys. JETP **35**, 908 (1972)].
[8] V. Kalantarov and O. Ladyzhenskaya, J. Sov. Math. **10**, 53 (1978).
[9] H. Fujita, J. Fac. Sci. Univ. Tokyo Sect. A Math. **16**, 105 (1966).
[10] M. I. Weinstein, Commun. Math. Phys. **87**, 567 (1983).
[11] E. A. Kuznetsov. J. J. Rasmussen, K. Rypdal, and S. K. Turitsyn (unpublished).
[12] A. C. Newell, in *Propagation in Systems Far from Equilibrium,* edited by J. E. Wesfreid, H. R. Brand, P. Manneville, G. Albinet, and N. Boccara (Springer, Berlin, 1988), p. 122, and references therein.
[13] A. C. Newell, *Pattern Formation and Pattern Recognition* (Springer-Verlag, Berlin, 1979); K. Stewartson and J. P. Stuart, J. Fluid Mech. **48**, 529 (1971).
[14] W. van Saarloos and P. C. Hohenberg, Physica D **56**, 303 (1992).
[15] W. Schöpf and L. Kramer, Phys. Rev. Lett. **66**, 2316 (1991).
[16] P. A. Belanger, J. Opt. Soc. Am. B **8**, 2077 (1991).
[17] L. M. Hocking, K. Stewartson, and J. P. Stuart, J. Fluid Mech. **51**, 705 (1972).
[18] S. K. Turitsyn, Phys. Rev. A **47**, R27 (1993).
[19] L. E. Payne and D. H. Sattinger, Isr. J. Math. **22**, 273 (1975).